Pergamon

PII: S0021-8928(97)00095-6

J. Appl. Maths Mechs, Vol. 61, No. 5, pp. 725–732, 1997

0021-8928/97 $24.00+0.00

MULTIPLE CAPTURE IN PONTRYAGIN'S EXAMPLE WITH PHASE CONSTRAINTS†

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(*Received* 20 *February* 1996)

Sufficient conditions for m-fold capture in Pontryagin's example [1] with many participants and with phase constraints on the state of the evader for identical dynamic and inertia possibilities of the players are derived. The phase constraint boundary here is not the "death line" for the evader. © 1998 Elsevier Science Ltd. All rights reserved.

This paper is related to the investigations described in [2–11].

1. STATEMENT OF THE PROBLEM

In the space $\mathbb{R}^k$ ($k \geqslant 2$), we consider an $n+1$-person differential game Γ: n pursuers $P_1, \ldots, P_n$ and an evader E with laws of motion

$$x_i^{(l)} + a_1 x_i^{(l-1)} + \ldots + a_l x_i = u_i, \quad u_i \in V \tag{1.1}$$

$$y^{(l)} + a_1 y^{(l-1)} + \ldots + a_l y = \upsilon, \quad \upsilon \in V \tag{1.2}$$

Here $x_i, y, u_i, \upsilon \in \mathbb{R}^k$, $a_1, \ldots, a_l \in \mathbb{R}^1$ and V is a convex compact set of $\mathbb{R}^k$. The initial conditions at $t = 0$ are

$$x_i^{(\alpha)}(0) = x_{i\alpha}^0, \quad y^{(\alpha)}(0) = y_\alpha^0, \quad \alpha = 0, \ldots, l-1 \tag{1.3}$$

where $x_{i0}^0 - y_0^0 \notin M_i$ for all i and M_i are convex compact sets of $\mathbb{R}^k$. Here and everywhere below $i = 1, \ldots, n$. It is also assumed that the evader E does not leave the convex set

$$D = \{y:\ y \in \mathbb{R}^k, (p_j, y) \leqslant \mu_j, \quad j = 1, \ldots, r\}$$

where $p_1, \ldots, p_r$ are unit vectors of $\mathbb{R}^k$ and $\mu_1, \ldots, \mu_r$ are real numbers such that $\operatorname{Int} D \neq 0$.

Definition 1. We shall say that m-fold capture occurs in the game Γ if there are the following: a time $T > 0$ and measurable functions $u_i(t) = u_i(t, x_{i\alpha}^0, y_\alpha^0, \upsilon(\cdot)) \in V$ and for any measurable function $\upsilon(t)$, $\upsilon(t) \in V$, $y(t) \in F$, $t \in [0\ T]$ there are times $\tau_1, \ldots, \tau_m \in [0, T]$ and pairwise different indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$ such that $x_{i\alpha}(\tau_\alpha) - y(\tau_\alpha) \in M_{i\alpha}$ for all $\alpha = 1, \ldots, m$.

It is assumed that $n \geqslant m$.

2. AUXILIARY ASSERTIONS

Lemma 1. Suppose the function

$$g(t) = \sum_{j=1}^{s} \exp(\lambda_j t) P_j(t) + \sum_{\alpha=1}^{q} \exp(\mu_\alpha t)(Q_\alpha(t)\cos \nu_\alpha t + R_\alpha(t)\sin \nu_\alpha t)$$

$$(\lambda_j, \mu_\alpha, \nu_\alpha \in R^1; \quad \lambda_1 < \lambda_2 < \ldots < \lambda_s, \quad \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_q)$$

where λ_j are pairwise different and P_j, Q_α, R_α are polynomials, is such that $g(t) \geqslant 0$ for all $t \geqslant 0$ and $g(t) \not\equiv 0$.

†*Prikl. Mat. Mekh.* Vol. 61, No. 5, pp. 747–754, 1997.

Then

1. $\mu_q \leqslant \lambda_s$;
2. If $\mu_q = \lambda_s$, then $\deg Q_p(t) \leqslant \deg P_s(t)$, $\deg R_p(t) \leqslant \deg P_s(t)$, for all $p \in I = \{p\colon \mu_p = \lambda_s\}$.

Instead of systems (1.1) and (1.2) we will consider the system

$$z_i^{(l)} + a_1 z_i^{(l-1)} + \ldots + a_l z_i = u_i - v\,, \quad u_i,\ v \in V \tag{2.1}$$
$$z_i(0) = z_{i0}^0 = x_{i0}^0 - y_0^0, \ldots, z_i^{(l-1)}(0) = z_{il-1}^0 = x_{il-1}^0 - y_{l-1}^0$$

We will denote by $\varphi_p(t)$ $(p = 0, 1, \ldots, l-1)$ solutions of the equation

$$w^{(l)} + a_1 w^{(l-1)} + \ldots + a_l w = 0$$

with initial conditions

$$w(0) = 0, \ldots, w^{(p-1)}(0) = 0, \quad w^{(p)}(0) = 1, \quad w^{(p+1)}(0) = 0, \ldots, w^{(l-1)}(0) = 0$$

Assumption 1. All the roots of the characteristic equation

$$\lambda^l + a_1 \lambda^{l-1} + \ldots + a_l = 0 \tag{2.2}$$

have non-positive real parts.

Assumption 2. The inequality $\varphi_{l-1}(t) \geqslant 0$ holds for all $t \geqslant 0$.

Note that Assumption 2 is satisfied if Eq. (2.2) has only real roots.

Assumption 2 and Lemma 1 imply that Eq. (2.2) has at least one real root. Let $\lambda_1, \ldots, \lambda_s$ $(\lambda_1 <, \ldots, < \lambda_s)$ denote the real roots, $\mu_1 \pm i\nu_1, \ldots, \mu_q \pm i\nu_q$ $(\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_q)$ the complex roots of Eq. (2.2), k_s the multiplicity of λ_s and m_α the multiplicity of the root $\mu_\alpha \pm i\nu_\alpha$. By the Assumption 2, $\mu_q \leqslant \lambda_s$. Further, let

$$\xi_i(t) = \varphi_0(t) z_{i0}^0 + \varphi_1(t) z_{i1}^0 + \ldots \varphi_{l-1}(t) z_{il-1}^0$$
$$\eta(t) = \varphi_0(t) y_0^0 + \varphi_1(t) y_1^0 \ldots \varphi_{l-1}(t) y_{l-1}^0$$

Then $\varphi_{l-1}(t)$, $\xi_i(t)$, $\eta(t)$ can be represented in the form

$$\varphi_{l-1}(t) = \sum_{j=1}^{s} \exp(\lambda_j t) P_{l-1j}(t) + \sum_{\alpha=1}^{q} \exp(\mu_\alpha t)(Q_{l-1\alpha}(t)\cos\nu_\alpha t + R_{l-1\alpha}(t)\sin\nu_\alpha t)$$
$$\xi_i(t) = \sum_{j=1}^{s} \exp(\lambda_j t) P_{ij}^1(t) + \sum_{\alpha=1}^{q} \exp(\mu_\alpha t)(Q_{i\alpha}^1(t)\cos\nu_\alpha t + R_{i\alpha}^1(t)\sin\nu_\alpha t)$$
$$\eta(t) = \sum_{j=1}^{s} \exp(\lambda_j t) P_j^2(t) + \sum_{\alpha=1}^{q} \exp(\mu_\alpha t)(Q_\alpha^2(t)\cos\nu_\alpha t + R_\alpha^2(t)\sin\nu_\alpha t)$$

We will assume that $\xi_i(t) \notin M_i$ for all i and $t > 0$, for if $\xi_\alpha(t) \in M_\alpha$ for some α and t, the pursuer P_α will catch the evader E, assuming $u_\alpha(t) = \upsilon(t)$, and we can then consider the problem of $(m-1)$-fold capture. We also assume that $P_{si}^1(t) \not\equiv 0$ for all i, for otherwise the pursuers initially endeavour to satisfy the given condition.

Let γ_i denote the degree of the polynomial $P_{si}^1(t)$, γ_0 the degree of the polynomial $P_s^2(t)$ and γ the degree of the polynomial $P_{sl-1}(t)$. It can be assumed that $\gamma_i = \gamma$ for all i, for otherwise pursuers P_i initially strive to satisfy the given condition.

Lemma 2. $\gamma = k_s - 1$.

Assumption 3. $m_\alpha < k_s$ for all $\alpha \in I = \{\alpha \mid \mu_\alpha = \lambda_s\}$.

Lemma 3. Suppose that Assumptions 1–3 are satisfied. Then for any $T > 0$, there are a constant $c > 0$ and a function $R(t)$, $\lim_{t\to\infty} R(t) = 0$ such that

$$\int_0^T \varphi_{l-1}(t-\tau)d\tau = c\exp(\lambda_s t)t^\gamma(1+R(t)) \quad (t>T)$$

Lemma 4. Assumptions 1–3 are satisfied.

1. Let $\lambda_s < 0$. Then there is a constant $a > 0$ for which

$$\lim_{t\to\infty}\int_T^t \varphi_{l-1}(t-\tau)d\tau = a \quad \forall T>0$$

2. Let $\lambda_s = 0$. Then for any $T > 0$ there are the following: a constant $c_1 > 0$ and a function $R_1(t)$, $\lim_{t\to\infty} R_1(t) = 0$ for which

$$\int_0^T \varphi_{l-1}(t-\tau)d\tau = c_1 t^\gamma(1+R_1(t))$$

We now define the function λ: $\mathrm{comp}(\mathbb{R}^k)\times V \to \mathbb{R}$

$$\lambda(A,v) = \sup\{\lambda \mid \lambda \geqslant 0, -\lambda A \cap (V-v) \neq 0\}$$

Here $\mathrm{comp}(\mathbb{R}^k)$ is the space of convex compact subsets of $\mathbb{R}^k$ with a Hausdorff metric. Suppose further that

$$z_i^0 = \lim_{t\to\infty} P_{si}^1/t^\gamma, \quad I=\{n+1,\dots,n+r\}$$

$$d =: \max\{\|v\|,\ v\in V\}$$

$$b =: \begin{cases} -1/a, & \text{if} \quad \lambda_s<0 \\ 0, & \text{if} \quad \lambda_s = 0 \end{cases}, \quad M_i^1 = \begin{cases} z_i^0 - M_i, & \text{if} \quad \lambda_s=0,\ k_s=1 \\ z_i^0 & \text{otherwise} \end{cases}$$

$$\Omega =: \{\{i_1,\dots,i_m\} : \{i_1,\dots,i_m\}\subset\{1,\dots,n\},\ i_1,\dots,i_m \quad \text{differ pairwise}\}$$

$$\lambda_j(v) = (p_{j-n}, v) + b\mu_{j-n}, \quad j\in I, \quad \xi_i^1(t) = \exp(-\lambda_s t)\xi_i(t)$$

$$\delta =: \inf_{v\in V}\max\{\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha(A_\alpha, v),\ \max_{j\in I}\lambda_j(v)\}$$

$$\delta_0 = \inf_{v\in V}\max\{\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha(M_\alpha^1, v),\ \max_{j\in I}\lambda_j(v)\}$$

$$V_1 = \{v : v\in V, \max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha(M_\alpha^1, v) = 0\}$$

Lemma 5. Let $A_1,\dots,A_n$ be convex compact sets such that $0\notin A_i$, $\delta > 0$ and let the functions $\lambda_i(A_i, \upsilon)$ be continuous at all points (A_i, υ), where $\lambda_i(A_i, \upsilon) > 0$. Then for any continuous multivalued mappings $B_i(t): [0,\infty)\to\mathrm{comp}(\mathbb{R}^k)$ for which $\lim_{t\to\infty} B_i(t) = A_i$ (in a Hausdorff metric), there is a time T_0 for which

$$\delta(t) = \inf_{v\in V}\max\{\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha(B_\alpha(t), v),\ \max_{j\in I}\lambda_j(v)\} \geqslant \frac{\delta}{2} \quad \forall t>T_0$$

Assumption 4. The condition $0\notin M_i^1$ holds and the functions λ_i are continuous at all points (M_i^1, υ) for which $\lambda_i(M_i^1, \upsilon) > 0$.

3. SUFFICIENT CONDITIONS FOR CAPTURE

Lemma 6. Suppose that for some game Γ Assumptions 1–4 hold and $\lambda_s < 0$, $0\in D$, $\delta_0 > 0$, $r = 1$. Then there is a time $T_0 > 0$ such that for any admissible function $\upsilon(\cdot)$, there is a set $\Lambda\in\Omega$ for which

$$1-\exp(-\lambda_s T_0)\int_0^{T_0}\varphi_{l-1}(T_0-\tau)\lambda_\alpha(\xi_\alpha^1(T_0), v(\tau))d\tau \leqslant 0 \quad \forall\alpha\in\Lambda$$

Proof. Let T be any number and let $\upsilon(t)$, $t\in[0,T]$ be an admissible function (that is, $y(t)\in D$ for all $t\in[0,T]$).

We will define the continuous functions

$$h_i(t)=1-\exp(-\lambda_s t)\int_0^t \varphi_{l-1}(t-\tau)\lambda_i(\xi_i^1(T),v(\tau))d\tau,\quad h_i(0)=1 \tag{3.1}$$

$$\sum_{\Lambda\in\Omega}\max_{\alpha\in\Lambda} h_\alpha(T)\leqslant C_n^m-\exp(-\lambda_s T)\int_0^T \varphi_{l-1}(T-\tau)\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha(\xi_\alpha^1(T),\,v(\tau))d\tau$$

Since $\xi_\alpha^1(T)/T^\gamma \to z_\alpha^0$ as $t\to\infty$, by virtue of Lemma 5 and Assumption 4, there is a time T_1 for which

$$\inf_v \max\left\{\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha\left(\frac{\xi_\alpha^1(t)}{t^\gamma},\,v\right),\quad \lambda_{n+1}(v)\right\}\geqslant\delta=\frac{\delta_0}{2}\quad \forall t>T_1$$

Since $y(t)\in D$ we have $(p_1,y(t))\leqslant\mu_1$. The last inequality is equivalent to the following

$$\int_{T_1}^t \varphi_{l-1}(t-\tau)(p_1,v(\tau))d\tau\leqslant\mu(t)=-(p_1,\eta(t))+\mu_1-\int_0^{T_1}\varphi_{l-1}(t-\tau)(p_1,v(\tau))d\tau$$

We will define the two sets $\Delta_1(t),\Delta_2(t)\subset[T_1,t]$, $(t>T_1)$ as follows:

$$\Delta_1(t)=\{\tau\mid\tau\in[T_1,t],\ (p_1,v(\tau))<\delta-b\mu_1=\delta_1\}$$
$$\Delta_2(t)=\{\tau\mid\tau\in[T_1,t],\ (p_1,v(\tau))\geqslant\delta_1\}$$

Then

$$G_1+G_2=f(t),\quad -dG_1+\delta_1G_2\leqslant\mu(t)$$

where

$$G_{1,2}=\int_{\Delta_{1,2}(t)}\varphi_{l-1}(t-\tau)d\tau,\quad f(t)=\int_{T_1}^t\varphi_{l-1}(t-\tau)d\tau$$

It follows from the last two relations that

$$G_1\geqslant(\delta_1 f(t)-\mu(t))/(d+\delta_1) \tag{3.2}$$

Assuming that $T>T_1$, from inequality (3.1) we obtain

$$\sum_{\Lambda\in\Omega}\max_{\alpha\in\Lambda}h_\alpha(T)\leqslant C_n^m-\exp(-\lambda_s T)\int_{\Delta_1(T)}\varphi_{l-1}(T-\tau)\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha(\xi_\alpha^1(T),v(\tau))d\tau \tag{3.3}$$

Since $\lambda_i(\xi^1{}_i(T),v)T^\gamma=\lambda_i(\xi^1{}_i(T)/T^\gamma,v)$

$$\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha(\xi_\alpha^1(T),\,v(\tau))=\frac{1}{T^\gamma}\max_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}\lambda_\alpha\left(\frac{\xi_\alpha^1(T)}{T^\gamma},\,v(\tau)\right)\geqslant\frac{\delta}{T^\gamma}\quad\forall\tau\in\Delta_1(T) \tag{3.4}$$

Thus from (3.3), taking account of (3.4) and (3.2), we obtain

$$\sum_{\Lambda\in\Omega}\min_{\alpha\in\Lambda}h_\alpha(T)\leqslant C_n^m-\frac{\exp(-\lambda_s T)\delta[\delta_1 f(T)-\mu(T)]}{T^\gamma(d+\delta_1)}=g(T)$$

From the relation

$$\frac{\exp(-\lambda_s T)\eta(T)}{T^\gamma}=\frac{P_s^2(T)}{T^\gamma}+\sum_{j=1}^{s-1}\exp(\lambda_j-\lambda_s)\frac{P_j^2(T)}{T^\gamma}+$$
$$+\sum_{\alpha=1}^{q}\exp((\mu_\alpha-\lambda_s)T)\left(\frac{Q_\alpha^2(T)}{T^\gamma}\cos\nu_\alpha T+\frac{R_\alpha^2(T)}{T^\gamma}\sin\nu_\alpha T\right)$$

the condition $\gamma_0\leqslant\gamma$ and Assumption 3, we see that the quantity $\|\exp(-\lambda_s T)\eta(T)/T^\gamma\|$ is bounded in $[T_1,\infty)$. Thus the quantity $\|\exp(-\lambda_s T)(p_1, \eta(T))/T^{\gamma}\|$ will also be bounded in $[T_1, \infty)$. It follows from Lemma 3 that the quantity

$$\frac{\exp(-\lambda_s T)}{T^{\gamma}} \int_0^{T_1} \varphi_{l-1}(T-\tau)d\tau$$

is bounded in $[T_1, \infty)$. From Lemma 4 it follows that

$$(\delta_1 f(t) - \mu_1) = (\delta - b\mu_1)f(t) - \mu_1 \to a\delta \quad \text{as} \quad t \to \infty$$

Thus $\lim g(T) = -\infty$ as $t \to \infty$. This means that there is a time T_0 which satisfies the condition of the lemma.

Let

$$V(t) = \{v_t(\cdot): v(\tau) \in V, \;\; y(\tau) \in D, \;\; \tau \in [0,t]\},$$

$$T(z_0) = \min\left\{t : t \geqslant 0, \quad \inf_{v_t(\cdot) \in V(t)} \max_{\Lambda \in \Omega} \min_{\alpha \in \Lambda} \int_0^t \exp(-\lambda_s t)\varphi_{l-1}(t-\tau)\lambda_\alpha(\xi_\alpha^1(t), v(\tau))d\tau \geqslant 1\right\}$$

Theorem 1. Suppose that for the game Γ Assumptions 1–4 are satisfied, $\lambda_s < 0$, $\delta_0 > 0$, $0 \in D$, $M_i = \{0\}$ and at least one of the following two conditions holds

$$\text{(a) } r = 1; \;\; \text{(b) } \min_{v \in \operatorname{co} V_1} \max_j ((p_j, v) + b\mu_j) > 0$$

Then the game Γ involves m-fold capture.

Proof. Suppose condition *a* holds. By Lemma 6 we have $T = T(z_0) < \infty$. Let $v(\tau)$ $(0 \leqslant \tau \leqslant T = T(z_0))$ be any admissible control of the evader E. There is a time $T_1 \in (0, T]$ which is a root of the function

$$h(t) = 1 - \max_{\Lambda \in \Omega} \min_{\alpha \in \Lambda} H_\alpha(t)$$

$$\left(H_\alpha(t) = \exp(-\lambda_s T) \int_0^t \varphi_{l-1}(T-\tau)\lambda_\alpha(\xi_\alpha^1(T), v(\tau))d\tau\right)$$

and also a set $\Lambda_0 \in \Omega$ such that $1 - H_\alpha(T_1) \leqslant 0$ for all $\alpha \in \Lambda_0$. Thus, there are times $t_\alpha \leqslant T_1$, $\alpha \in \Lambda_0$ such that

$$1 - H_\alpha(t_\alpha) = 0 \tag{3.5}$$

For $i \notin \Lambda_0$ we denote times for which Eq. (3.5) holds and $t_i \leqslant T_1$ by t_i.
We will fix the controls of pursuers P_i, putting

$$u_i(t) = v(t) - \lambda_i(\xi_i^1(T), v(t))\xi_i^1(T), \quad t \in [0, \min\{t_i, T_1\}]$$
$$u_i(t) = v(t), \quad t \in (\min\{t_i, T_1\}, T]$$

Then for all $\alpha \in \Lambda_0$

$$\exp(-\lambda_s T)z_\alpha(T) = \xi_\alpha^1(T) + \exp(-\lambda_s T)\int_0^T \varphi_{l-1}(T-\tau)(u_\alpha(\tau) - v(\tau))d\tau =$$

$$= \xi_\alpha^1(T)(1 - H_\alpha(t_\alpha))$$

From (3.5) we find that $\exp(-\lambda_s T)z_\alpha(T) = 0$ for all $\alpha \in \Lambda_0$. It follows that $z_\alpha(T) = 0$ for all $\alpha \in \Lambda_0$, and the theorem is proved in the case when $r = 1$.

Now suppose that condition *b* of the theorem is satisfied. Then by the theorem of Bohnenblust *et al.* [12] there are numbers $\gamma_j \geqslant 0$, $\gamma_1 + \ldots + \gamma_r = 1$ such that

$$\inf_{v \in \operatorname{co} V_1} \sum_{j=1}^{r} \gamma_j((p_j, v) + b\mu_j) > 0$$

Putting $p = \gamma_1 p_1 + \ldots + \gamma_r p_r$, $\mu = \gamma_1\mu_1 + \ldots + \gamma_r\mu_r$, $D_1 = \{y : y \in \mathbb{R}^k, (p, y) \leqslant \mu\}$, we have

$$\inf_{v \in V} \max\left\{\max_{\Lambda\in\Omega} \min_{\alpha\in\Lambda} \lambda_\alpha(z_\alpha^0, v),\ (p, v) + b\mu\right\} > 0$$

Thus, the problem of m-fold capture with phase constraints D_1 is solvable. Since $D \subset D_1$, the original problem on m-fold capture will also be solvable. This proves the theorem.

Corollary. Suppose that for the game Γ Assumptions 1–3 hold, $\lambda_s < 0$, $V = D_1(0)$, $\mu_j = 0, j = 1, \ldots, r$, $n \geqslant m + k - 1$ and

$$0 \in \operatorname{Int} \bigcap_{\Lambda(n-m+1)} \operatorname{co}\left\{\bigcup_{i\in\Lambda(n-m+1)} z_i^0, p_1, \ldots, p_r\right\} \tag{3.6}$$

Then m-fold capture occurs in the game Γ.

Lemma 7. Suppose that for the game Γ Assumptions 1–4 are satisfied, $\lambda_s = 0$, $\delta_0 > 0$, $r = 1$. Then there is a time T_0 such that, for each admissible function $v(t)$, there is a set $\Lambda \in \Omega$ for which

$$1 - \int_0^{T_0} \varphi_{l-1}(T_0 - \tau)\lambda_\alpha(\xi_\alpha(T_0) - M_\alpha, v(\tau))d\tau \leqslant 0 \quad \forall\alpha \in \Lambda$$

The proof is similar to that of Lemma 6.

Theorem 2. Suppose that for the game Γ Assumptions 1–4 hold, $\lambda_s = 0$, $\delta_0 > 0$, and at least one of the following two conditions holds

$$\text{(a) } r = 1; \quad \text{(b) } \min_{v \in \operatorname{co} V_1} \max_j (p_j, v) > 0$$

Then m-fold capture occurs in the game Γ.
The proof is similar to that of Theorem 1.

Corollary. Suppose that for the game Γ Assumptions 1–3 hold, $\lambda_s = 0$, $M_i = \{0\}$, $V = D_1(0)$, $n \geqslant k + m - 1$ and either condition (3.6) holds or D is a polytope.
Then in the game Γ m-fold capture occurs.

4. EXAMPLES

1. The laws of motion of the pursuers P_i and the evader E have the form

$$\dot{x}_i + ax_i = u_i, \quad x_i(0) = x_i^0, \quad u_i \in V, \quad a > 0$$
$$\dot{y} + ay = v, \quad y(0) = y_0^0, \quad v \in V$$

Let $M_i = \{0\}$, $0 \in D$. In that case

$$z_i^0 = x_i^0 - y_0^0, \quad b = -a, \quad \varphi_0(t) = \exp(-at)$$
$$\lambda_i(z_i^0, v) = ((z_i^0, v) + [(z_i^0, v)^2 + \| z_i^0 \|^2 (1 - \| v \|^2)]^{1/2}) / \| z_i^0 \|^2$$

Let

$$\delta = \min_v \max\left\{\max_{\Lambda\in\Omega} \min_{i\in\Lambda} \lambda_i(z_i^0, v),\ \max_{j\in I}((p_j, v) - a\mu_j)\right\}$$

Assertion 1. Let $\delta > 0$ and suppose that at least one of the following conditions holds

$$\text{(a) } r = 1; \quad \text{(b) } \min_{v \in \operatorname{co} V_1} \max_j ((p_j, v) - a\mu_j) > 0$$

Then m-fold capture occurs in the game Γ.

Assertion 2. Let $V = D_1(0)$, $\mu_j = 0, j = 1, \ldots, r, n \geqslant k + m - 1$ and suppose that condition (3.6) holds. Then m-fold capture occurs in the game Γ.

2. Systems (1.1) and (1.2) have the form

$$\begin{aligned} & x_i^{(4)} + 2x_i^{(3)} + \ddot{x}_i = u_i, \quad \|u_i\| \leqslant 1 \\ & x_i(0) = x_{i0}^0, \quad \dot{x}_i(0) = x_{i1}^0, \quad \ddot{x}_i(0) = x_{i2}^0, \quad x_i^{(3)}(0) = x_{i3}^0 \\ & y^{(4)} + 2y^{(3)} + \ddot{y} = v, \quad \|v\| \leqslant 1 \\ & y(0) = y_0^0, \quad \dot{y}(0) = y_1^0, \quad \ddot{y}(0) = y_2^0, \quad y^3(0) = y_3^0 \end{aligned} \tag{4.1}$$

In that case

$$\lambda_1 = -1, \; k_1 = 2, \; \lambda_2 = 0, \; k_2 = 2, \; \varphi_0(t) = 1, \; \varphi_1(t) = t$$
$$\varphi_2(t) = (3 + t)\exp(-t) + (2t - 3), \quad \varphi_3(t) = (2 + t)\exp(-t) + t - 2$$

We put

$$z_{iq}^0 = x_{iq}^0 - y_q^0, \quad z_i^0 = \begin{cases} z_{i1}^0 + 2z_{i2}^0 + z_{i3}^0, & z_{i1}^0 + 2z_{i2}^0 + z_{i3}^0 \neq 0 \\ z_{i0}^0 - 3z_{i2}^0 - 2z_{i3}^0, & z_{i1}^0 + 2z_{i2}^0 + z_{i3}^0 = 0 \end{cases}$$

We assume that $z_i^0 \neq 0$.

Assertion. Let $n \geqslant k + m - 1, M_i = \{0\}$ and suppose that condition (3.6) holds. Then m-fold capture occurs in the game Γ.

3. The form of systems (1.1) and (1.2) differs from (4.1) in the absence of the second term on the left-hand sides of the equations of motion of pursuers and evader. In that case

$$\lambda_1 = 0, \; k_1 = 2, \; \nu_1 = \pm i, \; m_1 = 1, \; \varphi_0(t) = 1, \; \varphi_1(t) = t$$
$$\varphi_2(t) = 1 - \cos t, \quad \varphi_3(t) = t - \sin t$$

Putting $z_{iq}^0 = x_{iq}^0 - y_q^0$, we have

$$\begin{aligned} \xi_i(t) &= \varphi_0(t) z_{i0}^0 + \varphi_1(t) z_{i1}^0 + \varphi_2(t) z_{i2}^0 + \varphi_3(t) z_{i3}^0 = \\ &= (z_{i0}^0 + z_{i2}^0) + t(z_{i1}^0 + z_{i3}^0) - (z_{i2}^0 \cos t + z_{i3}^0 \sin t) \end{aligned}$$

Let $z_i^0 = z_{i1}^0 + z_{i3}^0 \neq 0, M_i = \{0\}$.

Assertion. Let $n \geqslant k + m - 1$ and suppose that condition (3.6) holds. Then m-fold capture occurs in the game Γ.

This research, which is part of the "Russian Universities" programme (1.5.22), was supported financially by the Russian Foundation for Basic Research (94-01-00843a).

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Translated by R.L.